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BOUNDARY EFFECTS IN ELASTO-PLASTIC COSSERAT CONTINUA

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Abstract—The argument of discontinuous bifurcation is normally formulated for detecting spatial discontinuities in the interior of an infinite solution domain. This paper expands localization analysis to a finite domain in which the translatory notion of classical continua is enriched by independent microrotations in the spirit of Cosserat continua. The bifurcation condition for the interior is augmented by complementing conditions at the boundaries and at material interfaces.

In analogy to the equivalent propagation argument for Rayleigh surface waves and Stoneley waves within the traditional format of elasto-plasticity (see Needleman, A. and Ortiz, M. (1991). Effect of boundaries and interfaces on shear-band localization. *Int. J. Solids Struct.* **28**, 859–877) the current study develops the complementing propagation conditions for the micropolar Cosserat description of elasto-plasticity. To this end, their singularities are examined in terms of stationary Rayleigh surface waves and Stoneley waves at interior interfaces. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The formation of spatial discontinuities in rate-independent solids is commonly regarded as a local bifurcation problem (Rice, 1977). Aside from failure modes which are characterized by continuous deformation gradients, other failure modes may develop which exhibit a discontinuity in the field of deformation gradients across a singular surface preceding fracture. This discontinuity causes loss of ellipticity which results in strong mesh dependence in numerical computations.

To regularise the formation of discontinuity non-classical continuum theories like the micropolar theory by Cosserat (1909) and Eringen (1966) have been advocated. This theory introduces a characteristic length in a natural manner and describes the failure process in the vicinity of the localized deformation zone so that a non-local description results.

The analysis of discontinuities across the singular surface in micropolar continua leads to localization conditions which differ from those of classical continua in a basic sense. Not only the localization tensor has to be augmented to account for discontinuities of both the translatory velocity gradient as well as the rate of rotation gradient. There is also a second localization condition which has to be satisfied simultaneously and which plays an outstanding role to regularize localization. A formulation of a special case is presented in Dietsche *et al.* (1993) and Steinmann and Willam (1991).

From wave analysis of classical continua, it is well established that the onset of localization in an infinite continuum has an analogy with the formation of stationary body waves (see Hill (1962) and Rice (1977)). This paper focuses on the analogy between discontinuous bifurcation and stationary waves in micropolar continua with specific interest in the complementing conditions of stationary waves at the boundaries and at interior interfaces. To present the general form which exhibits the complete identity between the argument of stationary waves and spatial bifurcation conditions the basic formulas are reviewed. Starting point here is the integral consideration of a finite body which in a natural way leads to the differential equations in the interior and at the boundary.

2. CONSTITUTIVE EQUATIONS

Micropolar continua are characterized by rotational degrees of freedom $\boldsymbol{\omega} = [\omega_i]$, which are independent of the translation $\mathbf{u} = [u_i]$. The stress-tensor $\boldsymbol{\sigma} = [\sigma_{ij}]$ is non-symmetric because of the appearance of couple-stresses $\boldsymbol{\mu} = [\mu_{ij}]$. Consequently, the equations of motion including translatory as well as rotatory inertia effects are, in index notation,

$$\sigma_{ij,i} = \varrho u_{j,tt}, \mu_{ij,i} + e_{ikj}\sigma_{ik} = \Theta\omega_{j,tt} \tag{1}$$

where ϱ denotes the translatory mass density, Θ the rotatory mass density and $\mathbf{e} = [e_{ijk}]$ the permutation symbol. The kinematic description of independent microrotations leads to the loss of symmetry of the strain tensor $\boldsymbol{\varepsilon} = [\varepsilon_{ij}]$ and to the field of microcurvatures $\boldsymbol{\kappa} = [\kappa_{ij}]$ so that

$$\varepsilon_{ij} = u_{j,i} - e_{ijk}\omega_k, \quad \kappa_{ij} = \omega_{j,i} \tag{2}$$

(see Fig. 1). It has to be mentioned that, in this context, the couple-stress tensor $\boldsymbol{\mu}$ as well as the curvature tensor $\boldsymbol{\kappa}$ are both symmetric, otherwise their skewsymmetric parts would be undetermined.

To describe elasto-plastic behavior a non-symmetric extension of J_2-plasticity is adopted based on an associated flow rule (see also Besdo (1974)). The yield condition

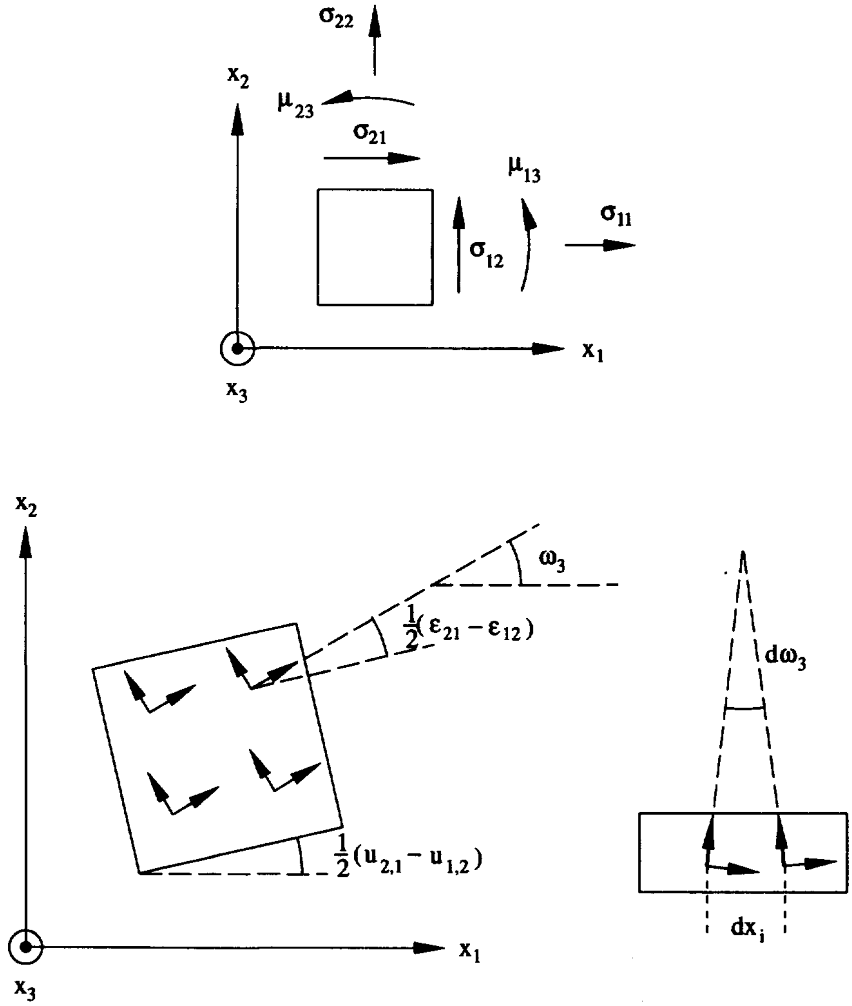

Fig. 1. Static (above) and kinematic (down) conditions.

couples the stress–strain and the couple-stress–curvature relations in terms of a single function of stress and of couple-stress $F = F(\boldsymbol{\sigma}, \boldsymbol{\mu}) = 0$.

Introducing the non-symmetric deviatoric stresses $\mathbf{s} = [s_{ij}]$ the yield condition

$$F(\boldsymbol{\sigma}, \boldsymbol{\mu}) = \left[3\left(\frac{1}{2}\mathbf{s}:\mathbf{s} + \frac{1}{2l_{c,s}^2}\boldsymbol{\mu}:\boldsymbol{\mu}\right)\right]^{1/2} - Y_0 = 0, \tag{3}$$

with Y_0 as yield stress, guarantees full coupling of non-symmetric stresses with symmetric couple-stresses. Expressing the two flow rules for the plastic strain and the plastic curvature rates in terms of a common plastic multiplier, the governing elasto-plastic rate equations may be cast into the partitioned form

$$\begin{aligned} \dot{\sigma}_{ij} &= E^{ep}_{ijkl}\dot{\varepsilon}_{kl} + D^{p,\kappa}_{ijkl}\dot{\kappa}_{kl}, \\ \dot{\mu}_{ij} &= D^{p,\varepsilon}_{ijkl}\dot{\varepsilon}_{kl} + C^{ep}_{ijkl}\dot{\kappa}_{kl} \end{aligned} \tag{4}$$

(see also Dietsche *et al.* (1993)), where $\mathbf{E}_{ep} = [E^{ep}_{ijkl}]$, $\mathbf{D}^{\kappa}_{p} = [D^{p,\kappa}_{ijkl}]$, $\mathbf{D}^{\varepsilon}_{p} = [D^{p,\varepsilon}_{ijkl}]$ and $\mathbf{C}_{ep} = [C^{ep}_{ijkl}]$ represent the elasto-plastic tangential material stiffness tensors:

$$\mathbf{E}_{ep} = \mathbf{E} - \frac{(\mathbf{E}:\mathbf{m}_\sigma)\otimes(\mathbf{n}_\sigma:\mathbf{E})}{E_p + \mathbf{n}_\sigma:\mathbf{E}:\mathbf{m}_\sigma + \mathbf{n}_\mu:\mathbf{C}:\mathbf{m}_\mu}, \tag{5}$$

$$\mathbf{D}^{\kappa}_{p} = -\frac{(\mathbf{E}:\mathbf{m}_\sigma)\otimes(\mathbf{n}_\mu:\mathbf{C})}{E_p + \mathbf{n}_\sigma:\mathbf{E}:\mathbf{m}_\sigma + \mathbf{n}_\mu:\mathbf{C}:\mathbf{m}_\mu}, \tag{6}$$

$$\mathbf{D}^{\varepsilon}_{p} = -\frac{(\mathbf{C}:\mathbf{m}_\mu)\otimes(\mathbf{n}_\sigma:\mathbf{E})}{E_p + \mathbf{n}_\sigma:\mathbf{E}:\mathbf{m}_\sigma + \mathbf{n}_\mu:\mathbf{C}:\mathbf{m}_\mu}, \tag{7}$$

$$\mathbf{C}_{ep} = \mathbf{C} - \frac{(\mathbf{C}:\mathbf{m}_\mu)\otimes(\mathbf{n}_\mu:\mathbf{C})}{E_p + \mathbf{n}_\sigma:\mathbf{E}:\mathbf{m}_\sigma + \mathbf{n}_\mu:\mathbf{C}:\mathbf{m}_\mu}. \tag{8}$$

According to (3), the two tensors

$$\mathbf{n}_\sigma = [n^\sigma_{ij}] = \frac{3}{2\sqrt{3J_2^*}}[s_{ij}], \quad \mathbf{n}_\mu = [n^\mu_{ij}] = \frac{3}{2l_c^2\sqrt{3J_2^*}}[\mu_{ij}] \tag{9}$$

including $J_2^* = (1/2)s_{ij}s_{ij} + (1/2l_c^2)\mu_{ij}\mu_{ij}$ denote the derivatives of the yield condition with regard to $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ and $\mathbf{m}_\sigma = [m^\sigma_{ij}]$, $\mathbf{m}_\mu = [m^\mu_{ij}]$ the derivatives of the plastic potential with regard to $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$.

The identity tensors $\mathbf{I}_2 = [\delta_{ij}]$, $\mathbf{I}_4^{sym} = 1/2([\delta_{ik}\,\delta_{jl}] + [\delta_{il}\,\delta_{jk}])$ and $\mathbf{I}_4^{skw} = 1/2([\delta_{ik}\,\delta_{jl}] - [\delta_{il}\,\delta_{jk}])$ define isotropic linear Cosserat elasticity in terms of the stiffness moduli $\mathbf{E} = [E_{ijkl}]$ and $\mathbf{C} = [C_{ijkl}]$ with

$$\mathbf{E} = \lambda\mathbf{I}_2\otimes\mathbf{I}_2 + 2G\mathbf{I}_4^{sym} + 2G_c\mathbf{I}_4^{skw}, \quad \mathbf{C} = 2Gl_{c,s}^2\mathbf{I}_2\otimes\mathbf{I}_2 \tag{10}$$

as functions of the Lamé constants λ, $\mu = G$, the Cosserat shear modulus G_c and the characteristic length $l_{c,s}$, while E_p denotes the hardening–softening-modulus of isotropic plasticity.

In the case of an associated flow rule ($\mathbf{m}_\sigma = \mathbf{n}_\sigma$, $\mathbf{m}_\mu = \mathbf{n}_\mu$), the plastic coupling operators $\mathbf{D}^{\varepsilon}_{p}$ and $\mathbf{D}^{\kappa}_{p}$ are related by

$$D^{p,\varepsilon}_{ijkl} = D^{p,\kappa}_{klij} \tag{11}$$

which designates symmetry, i.e., in matrix notation $\mathbf{D}^{\varepsilon}_{p} = (\mathbf{D}^{\kappa}_{p})^{\mathrm{T}}$. With the help of the identities

$$(\mathbf{I}_2 \otimes \mathbf{I}_2) : \mathbf{e} = \mathbf{I}^{sym}_4 : \mathbf{e} = \mathbf{0}_3, \quad \mathbf{A}^{sym} : \mathbf{e} = \mathbf{0}, \quad \mathbf{I}^{skw}_4 : \mathbf{e} = \mathbf{e} : \mathbf{I}^{skw}_4 = \mathbf{e} \tag{12}$$

(where $\mathbf{A}^{sym} = 1/2([A_{ij}] + [A_{ji}])$ is any symmetric tensor) the elasto-plastic tensor (5) extends to

$$\mathbf{E}_{ep} = \lambda \mathbf{I}_2 \otimes \mathbf{I}_2 + 2G\mathbf{I}^{sym}_4 + 2G_c\mathbf{I}^{skw}_4 - 4/b[G^2\mathbf{n}^{sym}_\sigma \otimes \mathbf{n}^{sym}_\sigma + GG_c(\mathbf{n}^{skw}_\sigma \otimes \mathbf{n}^{sym}_\sigma + \mathbf{n}^{sym}_\sigma \otimes \mathbf{n}^{skw}_\sigma) + G^2_c\mathbf{n}^{skw}_\sigma \otimes \mathbf{n}^{skw}_\sigma] \tag{13}$$

with

$$\begin{aligned} b &= E_p + 2G\mathbf{n}^{sym}_\sigma : \mathbf{n}^{sym}_\sigma + 2G_c\mathbf{n}^{skw}_\sigma : \mathbf{n}^{skw}_\sigma + Gl^2_{c,s}\mathbf{n}^{sym}_\mu : \mathbf{n}^{sym}_\mu, \\ \mathbf{n}^{sym}_\sigma &= [n^{\sigma,sym}_{ij}] = 1/2([n^\sigma_{ij}] + [n^\sigma_{ji}]), \quad \mathbf{n}^{skw}_\sigma = [n^{\sigma,skw}_{ij}] = 1/2([n^\sigma_{ij}] - [n^\sigma_{ji}]), \\ \mathbf{n}^{sym}_\mu &= [n^{\mu,sym}_{ij}] = 1/2([n^\mu_{ij}] + [n^\mu_{ji}]) \end{aligned} \tag{14}$$

for isotropic behavior. Consequently, the two plastic coupling material operators (6) and (7) can be explicitly evaluated as

$$\begin{aligned} \mathbf{D}^{\kappa}_p &= -(4Gl^2_{c,s}/b)(G\mathbf{n}^{sym}_\sigma + G_c\mathbf{n}^{skw}_\sigma) \otimes \mathbf{n}^{sym}_\mu, \\ \mathbf{D}^{\varepsilon}_p &= -(4Gl^2_{c,s}/b)\mathbf{n}^{sym}_\mu \otimes (G\mathbf{n}^{sym}_\sigma + G_c\mathbf{n}^{skw}_\sigma), \end{aligned} \tag{15}$$

and the elasto-plastic curvature material tensor (8) yields

$$\mathbf{C}_{ep} = 2G_c l^2_{c,s}\mathbf{I}_2 \otimes \mathbf{I}_2 - (4G^2 l^4_{c,s}/b)\mathbf{n}^{sym}_\mu \otimes \mathbf{n}^{sym}_\mu. \tag{16}$$

Note, that two characteristic lengths arise: besides the static length $l_{c,s}$ in $\mathbf{C}$, the rotary mass density Θ includes a dynamic characteristic length $l_{c,d}$ which is found in the definition of rotary inertia effects,

$$\Theta = \frac{1}{V}\int_m r^2\,\mathrm{d}m = l^2_{c,d}\varrho \tag{17}$$

with V as volume of the finite body, m as its mass and r as the radius. This dynamic length has no immediate geometrical meaning. It describes the influence of the rotary inertia in the smallest non-deformable region of the continuum represented by a mass point.

3. LOCALIZATION CONDITIONS

In the following, two localization conditions are developed. They can be derived from the assumption of a singularity surface $F_S(\mathbf{x}) = 0$ which separates the continuum into two

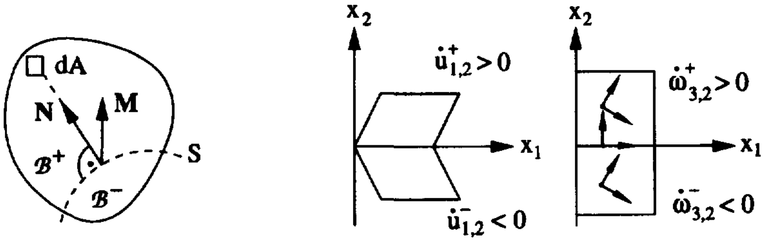

Fig. 2. Singularity surface (left) and assumed discontinuities of displacement and rotation gradient (right).

regions $\mathfrak{B}^+$ and $\mathfrak{B}^-$ with a common boundary S (see Fig. 2, left). Across this singularity surface, abrupt changes, respectively, jumps in the values of field variables, or their derivatives, occur, whereby outside of the surface, this field is assumed to be steady. Discontinuities in the form of jumps in the gradient fields of displacement and rotation rates across the singular surface are permitted (see, for the two dimensional case, Fig. 2, right). To establish the general form of the localization conditions let us review the basic balance laws.

First, the integral form of the balance of linear momentum is applied to a finite body containing a singularity surface:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}} \varrho\dot{\mathbf{u}}\,\mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^+} \varrho\dot{\mathbf{u}}\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^-} \varrho\dot{\mathbf{u}}\,\mathrm{d}V = \int_{\mathfrak{B}} \mathbf{b}_f\,\mathrm{d}V + \oint_{\partial\mathfrak{B}} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A \\ &= \int_{\mathfrak{B}^+} \mathbf{b}_f\,\mathrm{d}V + \int_{\mathfrak{B}^-} \mathbf{b}_f\,\mathrm{d}V + \int_{\partial\mathfrak{B}^+} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A + \int_{\partial\mathfrak{B}^-} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A \\ &\quad \underbrace{-\int_S [\![\mathbf{n}\cdot\boldsymbol{\sigma}]\!]\,\mathrm{d}A + \int_S \mathbf{n}\cdot\boldsymbol{\sigma}^+\,\mathrm{d}A - \int_S \mathbf{n}\cdot\boldsymbol{\sigma}^-\,\mathrm{d}A}_{=0} \end{aligned} \tag{18}$$

where $\dot{\mathbf{u}}$ denotes the translational velocity, $\mathbf{b}_f = [b_{f,i}]$ represents the forces per volume and $\mathbf{n} = [n_i]$ the surface direction vector. For the further considerations, it is necessary to augment the equation above by the zero value extension which includes the jump expression '$[\![\cdot]\!]$' describing the difference between the marked field variables on the plus and minus side of the singularity surface. The integral balance equations for each part of the separated body are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^+} \varrho\dot{\mathbf{u}}\,\mathrm{d}V &= \int_{\mathfrak{B}^+} \mathbf{b}_f\,\mathrm{d}V + \int_{\partial\mathfrak{B}^+} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A + \int_S \mathbf{n}\cdot\boldsymbol{\sigma}^+\,\mathrm{d}A \\ &= \int_{\mathfrak{B}^+} \mathbf{b}_f\,\mathrm{d}V + \int_{\mathfrak{B}^+} \operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}V = \mathbf{0}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^-} \varrho\dot{\mathbf{u}}\,\mathrm{d}V &= \int_{\mathfrak{B}^-} \mathbf{b}_f\,\mathrm{d}V + \int_{\partial\mathfrak{B}^-} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}A - \int_S \mathbf{n}\cdot\boldsymbol{\sigma}^-\,\mathrm{d}A \\ &= \int_{\mathfrak{B}^-} \mathbf{b}_f\,\mathrm{d}V + \int_{\mathfrak{B}^-} \operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}V = \mathbf{0}. \end{aligned} \tag{19}$$

Applying this equation to (18) with omitted body and inertia forces, all integrals vanish except the one with the jump expression. Further, regularity of the integrand leads to the usual equilibrium argument of Cauchy in rate form

$$[\![\mathbf{n}\cdot\dot{\boldsymbol{\sigma}}]\!] = \mathbf{0}, \tag{20}$$

which forms one part of the jump condition.

The second part of the static jump argument arises if the balance of angular momentum for the finite body

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}} \varrho\mathbf{r}\,\mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^+} \varrho\mathbf{r}\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^-} \varrho\mathbf{r}\,\mathrm{d}V = \int_{\mathfrak{B}} \mathbf{e}:\boldsymbol{\sigma}\,\mathrm{d}V + \oint_{\partial\mathfrak{B}} \mathbf{n}\cdot\boldsymbol{\mu}\,\mathrm{d}A \\
&= \int_{\mathfrak{B}^+} (\mathbf{e}:\boldsymbol{\sigma}+\mathbf{b}_m)\,\mathrm{d}V + \int_{\mathfrak{B}^-} (\mathbf{e}:\boldsymbol{\sigma}+\mathbf{b}_m)\,\mathrm{d}V \\
&\quad + \int_{\partial\mathfrak{B}^+} \mathbf{n}\cdot\boldsymbol{\mu}\,\mathrm{d}A + \int_{\partial\mathfrak{B}^-} \mathbf{n}\cdot\boldsymbol{\mu}\,\mathrm{d}A \\
&\quad \underbrace{- \int_S [\![\mathbf{n}\cdot\boldsymbol{\mu}]\!]\,\mathrm{d}A + \int_S \mathbf{n}\cdot\boldsymbol{\mu}^+\,\mathrm{d}A - \int_S \mathbf{n}\cdot\boldsymbol{\mu}^-\,\mathrm{d}A}_{=0}
\end{aligned} \tag{21}$$

is regarded whereby $\mathbf{r} = [r_i] = \dot{\omega} l_{c,d}^2$ denotes with (17) the rotational inertia and $\mathbf{b}_m = [b_{m,i}]$ the moments per volume. Again, augmentation of the angular balance law by the zero value extension is needed to formulate the jump condition. As the balance equation holds for both parts of the body, we can write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^+} \varrho\mathbf{r}\,\mathrm{d}V &= \int_{\mathfrak{B}^+} (\mathbf{e}:\boldsymbol{\sigma}+\mathbf{b}_m)\,\mathrm{d}V + \int_{\partial\mathfrak{B}^+} \mathbf{n}\cdot\boldsymbol{\mu}\,\mathrm{d}A + \int_S \mathbf{n}\cdot\boldsymbol{\mu}^+\,\mathrm{d}A \\
&= \int_{\mathfrak{B}^+} (\mathbf{e}:\boldsymbol{\sigma}+\mathbf{b}_m)\,\mathrm{d}V + \int_{\mathfrak{B}^+} \operatorname{div}\boldsymbol{\mu}\,\mathrm{d}V = \mathbf{0}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{B}^-} \varrho\mathbf{r}\,\mathrm{d}V &= \int_{\mathfrak{B}^-} (\mathbf{e}:\boldsymbol{\sigma}+\mathbf{b}_m)\,\mathrm{d}V + \int_{\partial\mathfrak{B}^-} \mathbf{n}\cdot\boldsymbol{\mu}\,\mathrm{d}A - \int_S \mathbf{n}\cdot\boldsymbol{\mu}^-\,\mathrm{d}A \\
&= \int_{\mathfrak{B}^-} (\mathbf{e}:\boldsymbol{\sigma}+\mathbf{b}_m)\,\mathrm{d}V + \int_{\mathfrak{B}^-} \operatorname{div}\boldsymbol{\mu}\,\mathrm{d}V = \mathbf{0}.
\end{aligned} \tag{23}$$

Omitting volume and inertia terms in (21) we are left with the jump expression in the rate form

$$[\![\mathbf{n}\cdot\dot{\boldsymbol{\mu}}]\!] = \mathbf{0} \tag{24}$$

as a second part of the jump condition.

To describe the stress and couple stress rate fields on both sides of the singularity surface with respect to the jumps of this fields across the surface it is possible to define bifurcated stress and couple stress fields such that

$$\begin{aligned}
\dot{\boldsymbol{\sigma}}^+(\mathbf{x}) &= \dot{\boldsymbol{\sigma}}(\mathbf{x}) + \zeta_\sigma(\mathbf{x}_s)[\![\dot{\boldsymbol{\sigma}}(\mathbf{x}_s)]\!], \quad \dot{\boldsymbol{\sigma}}^-(\mathbf{x}) = \dot{\boldsymbol{\sigma}}(\mathbf{x}) + (\zeta_\sigma(\mathbf{x}_s)-1)[\![\dot{\boldsymbol{\sigma}}(\mathbf{x}_s)]\!], \\
\dot{\boldsymbol{\mu}}^+(\mathbf{x}) &= \dot{\boldsymbol{\mu}}(\mathbf{x}) + \zeta_\mu(\mathbf{x}_s)[\![\dot{\boldsymbol{\mu}}(\mathbf{x}_s)]\!], \quad \dot{\boldsymbol{\mu}}^-(\mathbf{x}) = \dot{\boldsymbol{\mu}}(\mathbf{x}) + (\zeta_\mu(\mathbf{x}_s)-1)[\![\dot{\boldsymbol{\mu}}(\mathbf{x}_s)]\!].
\end{aligned} \tag{25}$$

The scalar fields $\zeta_\sigma(\mathbf{x}_S)$ and $\zeta_\mu(\mathbf{x}_S)$ describe the jump magnitude across the singularity surface, and $\mathbf{x}_S = \{\mathbf{x}\in\mathbb{R}^3 \mid F_S(\mathbf{x}) = 0\}$. Hence, it follows that $\mathbf{x}-\mathbf{x}_S = t\,\mathbf{grad}\,F_S$ represents the parametric form of the normal of the singularity surface. With this, the stress and couple stress state of each volume element outside the singularity surface is additively enhanced by the stress and couple stress states of a point on the singularity surface which is obtained by orthogonal projection from the volume element to the surface.

For the further evaluations only one side of the finite body is regarded. Neglecting inertia terms and forces per volume the balance equation for linear momentum can be rewritten with

$$\int_{\partial\mathfrak{B}^+} \mathbf{n}\cdot\dot{\boldsymbol{\sigma}}^+\,\mathrm{d}A - \int_S \mathbf{n}\cdot\dot{\boldsymbol{\sigma}}^{S,+}\,\mathrm{d}A = \int_{\mathfrak{B}^+} \operatorname{div}\dot{\boldsymbol{\sigma}}^+\,\mathrm{d}V = \int_{\mathfrak{B}^+} (\operatorname{div}\dot{\boldsymbol{\sigma}} + \zeta_\sigma \operatorname{div}[\![\dot{\boldsymbol{\sigma}}]\!])\,\mathrm{d}V = \mathbf{0} \quad (26)$$

and the balance equation for angular momentum without moments per volume yields

$$\int_{\mathfrak{B}^+} \mathbf{e}:\dot{\boldsymbol{\sigma}}^+\,\mathrm{d}V + \int_{\partial\mathfrak{B}^+} \mathbf{n}\cdot\dot{\boldsymbol{\mu}}^+\,\mathrm{d}A - \int_S \mathbf{n}\cdot\dot{\boldsymbol{\mu}}^{S,+}\,\mathrm{d}A = \int_{\mathfrak{B}^+} (\mathbf{e}:\dot{\boldsymbol{\sigma}}^+ + \operatorname{div}\dot{\boldsymbol{\mu}}^+)\,\mathrm{d}V$$

$$= \int_{\mathfrak{B}^+} (\mathbf{e}:\dot{\boldsymbol{\sigma}} + \operatorname{div}\dot{\boldsymbol{\mu}} + \zeta_\mu \mathbf{e}:[\![\dot{\boldsymbol{\sigma}}]\!] + \zeta_\mu \operatorname{div}[\![\dot{\boldsymbol{\mu}}]\!])\,\mathrm{d}V = \mathbf{0}. \quad (27)$$

For the integrand, regularity is assumed, which allows us to use the differential form. Applying $(19)_1$, (22) to (26), (27) a further jump condition,

$$\operatorname{div}[\![\dot{\boldsymbol{\sigma}}]\!] = \mathbf{0} \wedge \mathbf{e}:[\![\dot{\boldsymbol{\sigma}}]\!] + \operatorname{div}[\![\dot{\boldsymbol{\mu}}]\!] = \mathbf{0}, \quad (28)$$

is established. Obviously this result can also be obtained if the expressions related to $\mathfrak{B}^-$ are used.

In this context, the concept of a singularity surface of order two with respect to the translational degrees of freedom has to be enriched by a second spatially identic singularity surface of order one concerning the rotational degrees of freedom. This is enforced by the kinematic condition $(2)_1$ which requires interaction between the magnitudes of the translational field and the rotational field.

The material time derivative

$$\frac{\mathrm{d}\omega_i^+}{\mathrm{d}t} = \left(\frac{\partial\omega_i}{\partial t}\right)^+ + v^\omega n_j \omega_{i,j}^+ \quad (29)$$

differs from the local time derivative of the rotational degree of freedom $(\partial\omega_i/\partial t)^+$ by the convective term $v^\omega n_j \omega_{i,j}^+$. The difference for both sides of the finite body leads to the kinematic condition of compatibility (see Thomas, 1957)

$$\frac{\mathrm{d}[\![\omega_i]\!]}{\mathrm{d}t} = \left[\!\!\left[\frac{\partial\omega_i}{\partial t}\right]\!\!\right] + v^\omega n_j [\![\omega_{i,j}]\!] = [\![\dot{\omega}_i]\!] + v^\omega [\![n_j \omega_{i,j}]\!] \quad (30)$$

with $\mathbf{n} = [n_j]$ as the direction of the singularity surface and v^ω as the scalar normal component of the propagation rate. The field $\boldsymbol{\omega}$ is assumed to be continuous ($[\![\omega_i]\!] = 0$) so that

$$[\![\dot{\omega}_i]\!] = -v^\omega [\![n_j \omega_{i,j}]\!] \quad (31)$$

holds. This expression makes evident that a jump in velocity or rate is always connected with a jump in the deformation gradient.

The Maxwell compatibility conditions (see Maxwell, 1873) introduce, besides the direction vector $\mathbf{N} = [N_j]$, also the amplitude vector $\mathbf{M} = [M_i]$ which yields for a singularity surface of order two with respect to any vector field $\phi_i(x_i, t)$

$$[\![\dot{\phi}_{i,j}]\!] = -\dot{\gamma}^{\phi} M_i^{\phi} N_j, \tag{32}$$

and a singularity surface of order one is expressed by

$$[\![\dot{\phi}_i]\!] = -\dot{\gamma}\phi M_i^{\phi}. \tag{33}$$

With this definition first the jump conditions (20) and (24) are regarded (with $N_i = n_i$ and $\dot{\gamma}^{\omega} = \dot{\gamma}^{u} = \dot{\gamma}$). Combining them with the kinematic equations (2) and the tangential material law (4), the application of (32) and (33) results in

$$\begin{aligned}
[\![\mathbf{N}\cdot\dot{\boldsymbol{\sigma}}]\!] &= \mathbf{N}\cdot\mathbf{E}_{ep}:[\![\dot{\boldsymbol{\varepsilon}}]\!]+\mathbf{N}\cdot\mathbf{D}_p^{\kappa}:[\![\dot{\boldsymbol{\kappa}}]\!]\\
&= \mathbf{N}\cdot\mathbf{E}_{ep}:(\mathbf{N}\otimes\mathbf{M}^{u}-\mathbf{e}\cdot\mathbf{M}^{\omega})+\mathbf{N}\cdot\mathbf{D}_p^{\kappa}:(\mathbf{N}\otimes\mathbf{M}^{\omega}) = \mathbf{0},\\
[\![\mathbf{N}\cdot\dot{\boldsymbol{\mu}}]\!] &= \mathbf{N}\cdot\mathbf{D}_p^{\varepsilon}:[\![\dot{\boldsymbol{\varepsilon}}]\!]+\mathbf{N}\cdot\mathbf{C}_{ep}:[\![\dot{\boldsymbol{\kappa}}]\!]\\
&= \mathbf{N}\cdot\mathbf{D}_p^{\varepsilon}:(\mathbf{N}\otimes\mathbf{M}^{u}-\mathbf{e}\cdot\mathbf{M}^{\omega})+\mathbf{N}\cdot\mathbf{C}_{ep}:(\mathbf{N}\otimes\mathbf{M}^{\omega}) = \mathbf{0}.
\end{aligned} \tag{34}$$

With $\mathbf{M} = [\mathbf{M}^{u}, \mathbf{M}^{\omega}]$, this expression can be written by $\mathbf{Q}^{J1}\cdot\mathbf{M} = \mathbf{0}$ so that

$$\begin{bmatrix}\mathbf{Q}^{J,ee} & \mathbf{Q}^{J,ec}-\mathbf{N}\cdot\mathbf{E}_{ep}:\mathbf{e}\\ \mathbf{Q}^{J,ce} & \mathbf{Q}^{J,cc}-\mathbf{N}\cdot\mathbf{D}_p^{\varepsilon}:\mathbf{e}\end{bmatrix}\begin{bmatrix}\mathbf{M}^{u}\\ \mathbf{M}^{\omega}\end{bmatrix} = \mathbf{0}. \tag{35}$$

This denotes the first localization condition. To guarantee a solution of this homogeneous set of linear equations, the determinant of $\mathbf{Q}^{J1} = [Q_{ij}^{J1}]$ has to vanish. Here, the sub-operators are

$$\begin{aligned}
Q_{jl}^{J,ee} &= N_i E_{ijkl}^{ep} N_k, \quad Q_{jl}^{J,ec} = N_i D_{ijkl}^{p,\kappa} N_k,\\
Q_{jl}^{J,ce} &= N_i D_{ijkl}^{p,\varepsilon} N_k, \quad Q_{jl}^{J,cc} = N_i C_{ijkl}^{ep} N_k.
\end{aligned} \tag{36}$$

Considering the jump condition (28), a second condition for the onset of localization arises. If the kinematic equations (2) and the tangential material law (4) are used together with the Maxwell compatibility conditions (32) and (33), the jump conditions (28) render

$$\begin{aligned}
\operatorname{div}[\![\dot{\boldsymbol{\sigma}}]\!] &= \mathbf{N}\cdot\mathbf{E}_{ep}:(\mathbf{N}\otimes\mathbf{M}^{u}-\mathbf{e}\cdot\mathbf{M}^{\omega})+\mathbf{N}\cdot\mathbf{D}_p^{\kappa}:(\mathbf{N}\otimes\mathbf{M}^{\omega})\cdot\mathbf{N}\\
&= \mathbf{Q}^{J,ee}\cdot\mathbf{M}^{u}-\mathbf{N}\cdot\mathbf{E}_{ep}:\mathbf{e}\cdot\mathbf{M}^{\omega}+\mathbf{Q}^{J,ec}\cdot\mathbf{M}^{\omega} = \mathbf{0}
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
\mathbf{e}:[\![\dot{\boldsymbol{\sigma}}]\!]+\operatorname{div}[\![\dot{\boldsymbol{\mu}}]\!] &= \mathbf{e}:[\mathbf{E}_{ep}:(\mathbf{N}\otimes\mathbf{M}^{u}-\mathbf{e}\cdot\mathbf{M}^{\omega})+\mathbf{D}_p^{\kappa}:(\mathbf{N}\otimes\mathbf{M}^{\omega})]\\
&\quad +\mathbf{N}\cdot[\mathbf{D}_p^{\varepsilon}:(\mathbf{N}\otimes\mathbf{M}^{u}-\mathbf{e}\cdot\mathbf{M}^{\omega})+\mathbf{C}_{ep}:(\mathbf{N}\otimes\mathbf{M}^{\omega})]\\
&= \mathbf{e}:\mathbf{E}_{ep}\cdot\mathbf{N}\cdot\mathbf{M}^{u}-\mathbf{e}:\mathbf{E}_{ep}\cdot\mathbf{e}\cdot\mathbf{M}^{\omega}+\mathbf{e}:\mathbf{D}_p^{\kappa}\cdot\mathbf{N}\cdot\mathbf{M}^{\omega}\\
&\quad +\mathbf{Q}^{J,ce}\cdot\mathbf{M}^{u}-\mathbf{N}\cdot\mathbf{D}_p^{\varepsilon}:\mathbf{e}\cdot\mathbf{M}^{\omega}+\mathbf{Q}^{J,cc}\cdot\mathbf{M}^{\omega} = \mathbf{0}.
\end{aligned} \tag{38}$$

The matrix form of these two sets of homogeneous linear equations,

$$\begin{bmatrix} \mathbf{Q}^{J,ee} & \mathbf{Q}^{J,ec} - \mathbf{N}\cdot\mathbf{E}_{ep}:\mathbf{e} \\ \mathbf{Q}^{J,ce} + \mathbf{e}:\mathbf{E}_{ep}\cdot\mathbf{N} & \begin{matrix}\mathbf{Q}^{J,cc} - \mathbf{e}:\mathbf{E}_{ep}:\mathbf{e} \\ +\mathbf{e}:\mathbf{D}_p^{\kappa}\cdot\mathbf{N} - \mathbf{N}\cdot\mathbf{D}_p^{\varepsilon}:\mathbf{e}\end{matrix} \end{bmatrix} \begin{bmatrix} \mathbf{M}^u \\ \mathbf{M}^{\omega} \end{bmatrix} = \mathbf{0} \tag{39}$$

which in short notation reads $\mathbf{Q}^{J2}\cdot\mathbf{M} = 0$ can be compared with the first localization condition (35). The difference $(\mathbf{Q}^{J2} - \mathbf{Q}^{J1})\cdot\mathbf{M} = \mathbf{0}$ reduces (39) to

$$\mathbf{e}:[\mathbf{E}_{ep}:(\mathbf{N}\otimes\mathbf{M}^u - \mathbf{e}\cdot\mathbf{M}^{\omega}) + \mathbf{D}_p^{\kappa}:(\mathbf{N}\otimes\mathbf{M}^{\omega})] = \mathbf{0} \tag{40}$$

as the second localization condition. The explicit form of this with the material operators (5), (6) and (12) is

$$2G_c\dot{\gamma}\{\mathbf{e}:(\mathbf{N}\otimes\mathbf{M}^u) - 2\mathbf{M}^{\omega} - 2/b\,\mathbf{e}:\mathbf{n}_{\sigma}^{skw}[G_c\mathbf{n}_{\sigma}^{skw}:(\mathbf{N}\otimes\mathbf{M}^u - \mathbf{e}\cdot\mathbf{M}^{\omega}) + G((\mathbf{N}\otimes\mathbf{M}^u):\mathbf{n}_{\sigma}^{sym} + l_{c,s}^2\mathbf{n}_{\mu}^{sym}:(\mathbf{N}\otimes\mathbf{M}^{\omega}))]\} = \mathbf{0} \tag{41}$$

with b from (14).

Two special simplifications are of interest:

- If a symmetric stress state is assumed ($\mathbf{n}_{\sigma}^{skw} = 0$) (41) is simplified to

$$\mathbf{e}:(\mathbf{N}\otimes\mathbf{M}^u) = 2\mathbf{M}^{\omega}. \tag{42}$$

  Hence, the vector $\mathbf{M}^{\omega}$ is orthogonal to $\mathbf{N}$ and $\mathbf{M}^u$ whereby the angle between $\mathbf{N}$ and $\mathbf{M}^u$ is evaluated from the first localization condition (35).
- The second localization condition (40) can also be considered as the symmetry of the stress rate jumps $\mathbf{e}:[\![\dot{\boldsymbol{\sigma}}]\!] = \mathbf{0}$ which is a result of the material law with the operators (5)–(8) and the kinematic conditions. With the restriction that this symmetry statement is extended to the rate jumps of the two deformation fields, the condition

$$[\![\dot{\boldsymbol{\varepsilon}}]\!]^{skw} = 1/2\dot{\gamma}(\mathbf{N}\otimes\mathbf{M}^u - \mathbf{M}^u\otimes\mathbf{N}) - \dot{\gamma}\mathbf{e}\cdot\mathbf{M}^{\omega} = \mathbf{0} \tag{43}$$

  is equal to the expression

$$\mathbf{N}\otimes\mathbf{M}^u = \mathbf{e}\cdot\mathbf{M}^{\omega}. \tag{44}$$

  Multiplied by $1/2\mathbf{e}$ it represents a transformation of (42). Hence (41) with (42), (44) becomes

$$\mathbf{e}\cdot\mathbf{M}^{\omega}:\mathbf{n}_{\sigma}^{sym} + l_{c,s}^2\mathbf{n}_{\mu}^{sym}:(\mathbf{N}\otimes\mathbf{M}^{\omega}) = 0. \tag{45}$$

  Here, the first sum term is equal to zero because the permutation symbol acts on the symmetric tensor $\mathbf{n}_{\sigma}^{sym}$ so that the general solution of (45) is a vanishing $\mathbf{M}^{\omega}$ correlated with the coaxiality of $\mathbf{N}$ and $\mathbf{M}^u$ in (42).

However, the solution of the general form of the second localization condition (40) contains a vector system $\mathbf{M}^u$, $\mathbf{M}^{\omega}$, $\mathbf{N}$ whose angles clearly depend on the material constants as well as on the underlying elasto-plastic stress state.

It has to be emphasized that two aspects must be regarded to get the complete set of localization conditions. Besides the surface equations, the equations for the bulk material are also necessary. Nevertheless, in the case of classical continua, both points of view render identic statements so that no difference between them occurs and so only one form of the localization condition is obtained.

4. STATIONARY BODY WAVES

To extend the notion of a singularity surface of second order it is appropriate to think of surfaces with jumps in the second derivatives which are moving through an infinite continuum. The wave associated with a singularity surface of order two is the acceleration wave, where disturbances of the second time derivative of the underlying displacement field occur. This concept is correlated on the Fresnel–Hadamard theorem (see Hadamard, 1901) which requires that the amplitude $\mathbf{A} = [A_i]$ of an acceleration wave with the speed c^2 in the direction $\mathbf{N}$ has to be an eigenvector of the acoustic tensor $\mathbf{Q} = [Q_{ij}]$, that is

$$\mathbf{Q}\cdot\mathbf{A} = c^2 \varrho k_k k_k \mathbf{A}. \tag{46}$$

Vice versa stationary waves with $c^2 = 0$ are identical with the onset of localization because both assumptions are synonymous with the singularity of the tensor $\mathbf{Q}$.

The underlying differential equations in (1) which describe the motion of a disturbance through a continuum are, in general, hyperbolic as long as the wave speed c^2 is greater than zero. The transition to the limiting value $c^2 = 0$ causes, therefore, loss of hyperbolicity. So the loss of hyperbolicity and the loss of ellipticity describe analogous transitions of state.

Similar to the procedure in the previous chapter, two steps have to be carried out to get the complete set of stationary wave equations concerning a finite body: besides the conditions in an infinite volume, the equations at its surface are also of interest.

To analyse the propagation of waves in an infinite continuum the equations of motion and the kinematic conditions (1) and (2) are combined with the elasto-plastic rate eqns (4) into

$$\begin{aligned} E^{ep}_{ijkl}(u_{l,ki} - e_{klm}\omega_{m,i}) + D^{p,\kappa}_{ijkl}\omega_{l,ki} &= \varrho u_{j,tt}, \\ D^{p,\varepsilon}_{ijkl}(u_{l,ki} - e_{klm}\omega_{m,i}) + C^{ep}_{ijkl}\omega_{l,ki} + e_{ikj}[E^{ep}_{iklm}(u_{m,l} - e_{lmn}\omega_n) + D^{p,\kappa}_{iklm}\omega_{m,l}] &= \Theta\omega_{j,tt}. \end{aligned} \tag{47}$$

The complex waves provide solutions for the linear wave equations in terms of

$$u_j = A^u_j \exp[\mathrm{i}(k_k x_k - \sqrt{k_k k_k}ct)], \quad \omega_j = A^\omega_j \exp[\mathrm{i}(k_k x_k - \sqrt{k_k k_k}ct)] \tag{48}$$

where $\mathbf{k} = [k_k]$ designates the wave number, $\mathbf{A}^u = [A^u_j]$, $\mathbf{A}^\omega = [A^\omega_j]$ the wave amplitudes with complex components being omitted, and $\mathrm{i} = \sqrt{-1}$.

Additionally, the complex permutation symbol $\varepsilon = \mathrm{i}e$ has to be defined so that the equation

$$\varepsilon_{ijk} A^\omega_k \exp[\mathrm{i}(k_k x_k - \sqrt{k_k k_k}ct)] = \mathrm{i}\tfrac{1}{2}(A^u_j k_i - A^u_i k_j)\exp[\mathrm{i}(k_k x_k - \sqrt{k_k k_k}ct)] \tag{49}$$

is fulfilled which guarantees that the special case of a symmetric deformation tensor $\boldsymbol{\varepsilon}$ is also described in the imaginary space.

Introducing this solution into (47), the two equations are transformed into

$$\begin{aligned} Q^{B,ee}_{jl}A^u_l + \mathrm{i}E^{ep}_{ijkl}\varepsilon_{klm}A^\omega_m k_i + Q^{B,ec}_{jl}A^\omega_l &= \varrho c^2 A^u_j k_k k_k, \\ Q^{B,ce}_{jl}A^u_l + \mathrm{i}D^{p,\varepsilon}_{ijkl}\varepsilon_{klm}A^\omega_m k_i + Q^{B,cc}_{jl}A^\omega_l - \mathrm{i}\varepsilon_{ikj}E^{ep}_{iklm}A^u_m k_l & \\ + \varepsilon_{ikj}E^{ep}_{iklm}\varepsilon_{lmn}A^\omega_n - \mathrm{i}\varepsilon_{ikj}D^{p,\kappa}_{iklm}A^\omega_m k_l &= \Theta c^2 A^\omega_j k_k k_k \end{aligned} \tag{50}$$

which yields the wave velocity c^2. Obviously, (50) represents dispersive relations, and the wave speed is also a function of the wave number. However, this fact can be neglected in the further considerations because stationary waves (with $c^2 = 0$) are not influenced by variations of the wave number.

For stationary non-propagating waves, (50) is expressed by the matrix form $\mathbf{Q}^{B2}\cdot\mathbf{A} = \mathbf{0}$ (with $\mathbf{A} = [\mathbf{A}^u, \mathbf{A}^\omega]$) which is, in detail

$$\begin{bmatrix} Q_{jl}^{B,ee} & Q_{jl}^{B,ec} - k_i E_{ijkm}^{ep} e_{kml} \\ Q_{jl}^{B,ce} + e_{ikj} E_{ikml}^{ep} k_m & Q_{jl}^{B,cc} - e_{ikj} E_{iknm}^{ep} e_{nml} \\ & + e_{ikj} D_{ikml}^{p,\kappa} k_m + k_i D_{ijkm}^{p,\varepsilon} e_{kml} \end{bmatrix} \begin{bmatrix} A_l^u \\ A_l^\omega \end{bmatrix} = 0_j. \tag{51}$$

The sub-operators of the acoustic tensor $\mathbf{Q}^{B2}$ are defined analogous to (36) by $\mathbf{Q}^{B,xx} = \mathbf{k} \cdot \mathbf{F}_{xx} \cdot \mathbf{k}$ where $\mathbf{F}_{xx} \in \{\mathbf{E}_{ep}, \mathbf{D}_p^\kappa, \mathbf{D}_p^\varepsilon, \mathbf{C}_{ep}\}$.

Next, in the balance equations at the surface (with $N_i = k_i$)

$$k_i E_{ijkl}^{ep}(u_{l,k} - e_{klm}\omega_m) + k_i D_{ijkl}^{p,\kappa}\omega_{l,k} = 0_j,$$
$$k_i D_{ijkl}^{p,\varepsilon}(u_{l,k} - e_{klm}\omega_m) + k_i C_{ijkl}^{ep}\omega_{l,k} = 0_j \tag{52}$$

the complex solutions (48) are applied so that

$$\mathrm{i}k_i E_{ijkl}^{ep} k_k A_l^u + \mathrm{i}k_i D_{ijkl}^{p,\kappa} k_k A_l^\omega - k_i E_{ijkm}^{ep} \varepsilon_{kml} A_l^\omega = 0_j,$$
$$\mathrm{i}k_i D_{ijkl}^{p,\varepsilon} k_k A_l^u + \mathrm{i}k_i C_{ijkl}^{ep} k_k A_l^\omega - k_i D_{ijkm}^{p,\varepsilon} \varepsilon_{kml} A_l^\omega = 0_j \tag{53}$$

results. The sub-operators are the same as defined in (51) so that the set of linear homogeneous equations reads

$$\begin{bmatrix} Q_{jl}^{B,ee} & Q_{jl}^{B,ec} - k_i E_{ijkm}^{ep} e_{kml} \\ Q_{jl}^{B,ce} & Q_{jl}^{B,cc} - k_i D_{ijkm}^{p,\varepsilon} e_{kml} \end{bmatrix} \begin{bmatrix} A_l^u \\ A_l^\omega \end{bmatrix} = 0_j \tag{54}$$

or $\mathbf{Q}^{B1} \cdot \mathbf{A} = \mathbf{0}$. With the difference $(\mathbf{Q}^{B2} - \mathbf{Q}^{B1}) \cdot \mathbf{A} = \mathbf{0}$ in terms of

$$e_{ikj}[E_{ikml}^{ep}(k_m A_l^u - e_{mln} A_n^\omega) + D_{ikml}^{p,\kappa} k_m A_l^\omega] = 0_j, \tag{55}$$

the second condition for stationary waves is found. Applying the equalities $\mathbf{k} = \mathbf{N}$ and $\mathbf{A} = \mathbf{M}$, the two conditions for stationary waves (54), (55) are identical to the localization conditions (35) and (40).

Considering classical continuum formulations, the statement (46) may be transformed

$$\mathbf{A}^{\mathrm{T}} \cdot \mathbf{Q} \cdot \mathbf{A} = c^2 \varrho k_k k_k. \tag{56}$$

If $c^2 \to 0$, this is equivalent with the well known notion of 'loss of strong ellipticity'.

To develop this condition for Cosserat continua, both systems of linear equations (50) and (53) are used. The first one is cast into the form

$$\mathbf{Q}^{B2} \cdot \mathbf{A} = \varrho c^2 k_k k_k \mathbf{D} \cdot \mathbf{A} \tag{57}$$

with the matrix

$$\mathbf{D} = \begin{bmatrix} \delta_{ij} & 0 \\ 0 & \delta_{ij} l_{c,d}^2 \end{bmatrix}. \tag{58}$$

Here, the decomposition of the rotational inertia in (17) has been used. Multiplying $\mathbf{A}^{\mathrm{T}}$ on both sides, (57) becomes

$$\mathbf{A}^{\mathrm{T}} \cdot \mathbf{Q}^{B2} \cdot \mathbf{A} = \varrho c^2 k_k k_k \tag{59}$$

whereby $\mathbf{A}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{A} = A_i^u A_i^u + l_d^2 A_i^\omega A_i^\omega = 1$ is the underlying assumption.

The second set of homogeneous linear equations $\mathbf{A}^{\mathrm{T}} \cdot \mathbf{Q}^{B1} \cdot \mathbf{A} = 0$ yields

$$\mathbf{A}^u \cdot \mathbf{Q}^{B,ee} \cdot \mathbf{A}^u + \mathbf{A}^\omega \cdot \mathbf{Q}^{B,ce} \cdot \mathbf{A}^u + \mathbf{A}^u \cdot (\mathbf{Q}^{B,ec} - \mathbf{k} \cdot \mathbf{E}_{ep} : \mathbf{e}) \cdot \mathbf{A}^\omega + \mathbf{A}^\omega \cdot (\mathbf{Q}^{B,cc} - \mathbf{k} \cdot \mathbf{D}_p^\varepsilon : \mathbf{e}) \cdot \mathbf{A}^\omega = 0. \quad (60)$$

If this zero expression is used in the evaluation of (57) where $c^2 = 0$ the second statement

$$\mathbf{A}^\omega \cdot \mathbf{e} : [\mathbf{E}_{ep} : (\mathbf{k} \otimes \mathbf{A}^u - \mathbf{e} \cdot \mathbf{A}^\omega) + \mathbf{D}_p^\kappa : (\mathbf{k} \otimes \mathbf{A}^\omega)] = 0 \quad (61)$$

arises which obviously reflects (55).

5. STATIONARY RAYLEIGH WAVES

A special case of boundary condition is obtained if a free surface is considered which reveals damping in the direction to the bulk material. Hence, only waves parallel to this surface can propagate free, orthogonal to this surface the wave amplitudes decrease rapidly.

Assumed the damping influence acts along the x_1-direction, the complex wave solutions

$$u_j = A_j^u \exp [\mathrm{i}(k_k^R x_k - \sqrt{k_k^R k_k^R} ct)], \quad \omega_j = A_j^\omega \exp [\mathrm{i}(k_k^R x_k - \sqrt{k_k^R k_k^R} ct)] \quad (62)$$

are correlated with the wave number vector

$$\mathbf{k}^R = [k_k^R] = [\mathrm{i}\Im(k_1), \Re(k_2), \Re(k_3)]^{\mathrm{T}} = [\mathrm{i}v_1, v_2, v_3]^{\mathrm{T}} \quad (63)$$

where v_i represent real numbers. The additional condition for the complex solution (62) for exponential decay into the body requires that $\Im(k_1) = v_1 \geqslant 0$.

A traction free surface requires

$$t_j^\sigma = N_i \sigma_{ij} = 0_j, \quad t_j^\mu = N_i \mu_{ij} = 0_j \quad (64)$$

where N_i defines the normal vector component of the bounding surface. In Fig. 3, the surface $x_1 = 0$ with orthogonal damping in $x_1 \geqslant 0$ is shown. In case of stationary waves the direction vector $\mathbf{N}$ is replaced by the wave number vector $\mathbf{k}^R$. With this the boundary conditions are

$$k_i^R \sigma_{ij} = 0_j, \quad k_i^R \mu_{ij} = 0_j. \quad (65)$$

Together with the constitutive rate equations (4), the two set of linear homogeneous equations get the matrix form $\mathbf{Q}^{R1} \cdot \mathbf{A} = \mathbf{0}$ or

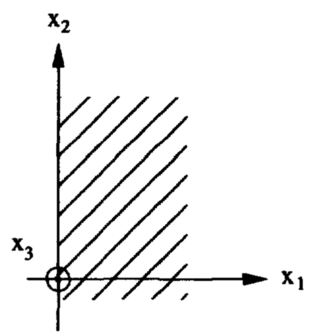

Fig. 3. Boundary conditions for Rayleigh waves.

$$\begin{bmatrix} \mathbf{Q}^{R,ee} & \mathbf{Q}^{R,ec}-\mathbf{k}^R\cdot\mathbf{E}_{ep}:\mathbf{e} \\ \mathbf{Q}^{R,ce} & \mathbf{Q}^{R,cc}-\mathbf{k}^R\cdot\mathbf{D}_p^{\varepsilon}:\mathbf{e} \end{bmatrix}\begin{bmatrix}\mathbf{A}^u \\ \mathbf{A}^{\omega}\end{bmatrix}=\mathbf{0} \tag{66}$$

where

$$\begin{aligned} Q_{jl}^{R,ee} &= k_i^R E_{ijkl}^{ep} k_k^R = v_i E_{ijkl}^{ep} v_k - v_1 E_{1j1l}^{ep} v_1 + \mathrm{i}(v_i E_{ij1l}^{ep} v_1 + v_1 E_{1jkl}^{ep} v_k) \\ Q_{jl}^{R,ec} &= k_i^R D_{ijkl}^{p,\kappa} k_k^R = v_i D_{ijkl}^{p,\kappa} v_k - v_1 D_{1j1l}^{p,\kappa} v_1 + \mathrm{i}(v_i D_{ij1l}^{p,\kappa} v_1 + v_1 D_{1jkl}^{p,\kappa} v_k) \\ Q_{jl}^{R,ce} &= k_i^R D_{ijkl}^{p,\varepsilon} k_k^R = v_i D_{ijkl}^{p,\varepsilon} v_k - v_1 D_{1j1l}^{p,\varepsilon} v_1 + \mathrm{i}(v_i D_{ij1l}^{p,\varepsilon} v_1 + v_1 D_{1jkl}^{p,\varepsilon} v_k) \\ Q_{jl}^{R,cc} &= k_i^R C_{ijkl}^{ep} k_k^R = v_i C_{ijkl}^{ep} v_k - v_1 C_{1j1l}^{ep} v_1 + \mathrm{i}(v_i C_{ij1l}^{ep} v_1 + v_1 C_{1jkl}^{ep} v_k). \end{aligned} \tag{67}$$

The complex wave number vector (63) applied to the wave propagation in the bulk material leads to $\mathbf{Q}^{R2}\cdot\mathbf{A}=\mathbf{0}$, respectively,

$$\begin{bmatrix} \mathbf{Q}^{R,ee} & \mathbf{Q}^{R,ec}-\mathbf{k}^R\cdot\mathbf{E}_{ep}:\mathbf{e} \\ \mathbf{Q}^{R,ce}+\mathbf{e}:\mathbf{E}_{ep}\cdot\mathbf{k}^R & \begin{matrix}\mathbf{Q}^{R,cc}-\mathbf{e}:\mathbf{E}_{ep}:\mathbf{e} \\ +\mathbf{e}:\mathbf{D}_p^{\kappa}\cdot\mathbf{k}^R-\mathbf{k}^R\cdot\mathbf{D}_p^{\varepsilon}:\mathbf{e}\end{matrix} \end{bmatrix}\begin{bmatrix}\mathbf{A}^u \\ \mathbf{A}^{\omega}\end{bmatrix}=\mathbf{0} \tag{68}$$

where the sub-operators are defined as above.

The difference of $(\mathbf{Q}^{R2}-\mathbf{Q}^{R1})\cdot\mathbf{A}=\mathbf{0}$,

$$\mathbf{e}:[\mathbf{E}_{ep}:(\mathbf{k}^R\otimes\mathbf{A}^u-\mathbf{e}\cdot\mathbf{A}^{\omega})+D_p^{\kappa}:(\mathbf{k}^R\otimes\mathbf{A}^{\omega})]=\mathbf{0} \tag{69}$$

is, beside (66), the second condition for the occurrence of stationary Rayleigh waves.

6. STATIONARY STONELEY WAVES

Next, the argument of stationary waves is extended to different materials. Assumed that two half spaces at $x_1=0$ are characterized with different material operators an interface is defined (see Fig. 4).

So, the complex solutions

$$\begin{aligned} u_j^{\pm} &= A_j^{u,\pm}\exp\left[\mathrm{i}\left(k_k^{S,\pm}x_k-\sqrt{k_k^{S,\pm}k_k^{S,\pm}}\,ct\right)\right], \\ \omega_j^{\pm} &= A_j^{\omega,\pm}\exp\left[\mathrm{i}\left(k_k^{S,\pm}x_k-\sqrt{k_k^{S,\pm}k_k^{S,\pm}}\,ct\right)\right] \end{aligned} \tag{70}$$

include the wave number vector

$$\mathbf{k}_{\pm}^S=[k_i^{S,\pm}]=[\mathrm{i}\Im(k_1^{\pm}),\Re(k_2^{\pm}),\Re(k_3^{\pm})]^{\mathrm{T}}=[\mathrm{i}v_1^{\pm},v_2^{\pm},v_3^{\pm}]^{\mathrm{T}}. \tag{71}$$

To enable decay into the body on both sides of the interface, it is necessary that $v_1^+>0$, $v_1^-<0$. Further, the compatibility conditions at the interface are

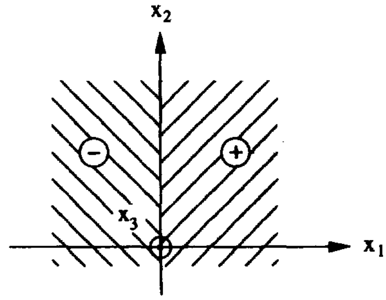

Fig. 4. Boundary conditions for Stoneley waves.

$$\mathbf{A}^u_+ = \mathbf{A}^u_- = \mathbf{A}^u, \quad \mathbf{A}^\omega_+ = \mathbf{A}^\omega_- = \mathbf{A}^\omega,$$
$$v_2^+ = v_2^- = v_2, \quad v_3^+ = v_3^- = v_3. \tag{72}$$

With this notation the boundary conditions can be written as

$$\mathbf{N}\cdot\boldsymbol{\sigma}^+ - \mathbf{N}\cdot\boldsymbol{\sigma}^- = \mathbf{0}, \quad \mathbf{N}\cdot\boldsymbol{\mu}^+ - \mathbf{N}\cdot\boldsymbol{\mu}^- = \mathbf{0}. \tag{73}$$

The direction vector $\mathbf{N}$ is replaced by $\mathbf{k}^S_\pm$ and the rate eqns (4) with the complex solutions (70), (72) are used so that the matrix form of (73) becomes $\mathbf{Q}^{S1}\cdot\mathbf{A} = \mathbf{0}$ or

$$\begin{bmatrix} \mathbf{Q}^{S,ee}_+ - \mathbf{Q}^{S,ee}_- & \begin{matrix} \mathbf{Q}^{S,ec}_+ - \mathbf{k}^S_+\cdot\mathbf{E}^+_{ep} : \mathbf{e} \\ -(\mathbf{Q}^{S,ec}_- - \mathbf{k}^S_-\cdot\mathbf{E}^-_{ep} : \mathbf{e}) \end{matrix} \\ \mathbf{Q}^{S,ce}_+ - \mathbf{Q}^{S,ce}_- & \begin{matrix} \mathbf{Q}^{S,cc}_+ - \mathbf{k}^S_+\cdot\mathbf{D}^{\varepsilon,+}_p : \mathbf{e} \\ -(\mathbf{Q}^{S,cc}_- - \mathbf{k}^S_-\cdot\mathbf{D}^{\varepsilon,-}_p : \mathbf{e}) \end{matrix} \end{bmatrix} \begin{bmatrix} \mathbf{A}^u \\ \mathbf{A}^\omega \end{bmatrix} = \mathbf{0}. \tag{74}$$

A similar procedure can be applied to both sides of the interface so that stationary waves are expressed in terms of $\mathbf{Q}^{S2}\cdot\mathbf{A} = \mathbf{0}$ so that

$$\begin{bmatrix} \mathbf{Q}^{S,ee}_+ - \mathbf{Q}^{S,ee}_- & \begin{matrix} \mathbf{Q}^{S,ec}_+ - \mathbf{k}^S_+\cdot\mathbf{E}^+_{ep} : \mathbf{e} \\ -(\mathbf{Q}^{S,ec}_- - \mathbf{k}^S_-\cdot\mathbf{E}^-_{ep} : \mathbf{e}) \end{matrix} \\ \begin{matrix} \mathbf{Q}^{S,ce}_+ + \mathbf{e}:\mathbf{E}^+_{ep}\cdot\mathbf{k}^S_+ \\ -(\mathbf{Q}^{S,ce}_- + \mathbf{e}:\mathbf{E}^-_{ep}\cdot\mathbf{k}^S_-) \end{matrix} & \begin{matrix} \mathbf{Q}^{S,cc}_+ - \mathbf{e}:\mathbf{E}^+_{ep}:\mathbf{e} \\ +\mathbf{e}:\mathbf{D}^{\kappa,+}_p\cdot\mathbf{k}^S_+ - \mathbf{k}^S_+\cdot\mathbf{D}^{\varepsilon,+}_p : \mathbf{e} \\ -(\mathbf{Q}^{S,cc}_- + \mathbf{e}:\mathbf{E}^-_{ep}:\mathbf{e} \\ +\mathbf{e}:\mathbf{D}^{\kappa,-}_p\cdot\mathbf{k}^S_- - \mathbf{k}^S_-\cdot\mathbf{D}^{\varepsilon,-}_p : \mathbf{e}) \end{matrix} \end{bmatrix} \begin{bmatrix} \mathbf{A}^u \\ \mathbf{A}^\omega \end{bmatrix} = \mathbf{0}. \tag{75}$$

The difference of $(\mathbf{Q}^{S2} - \mathbf{Q}^{S1})\cdot\mathbf{A} = \mathbf{0}$ exhibits besides (74) the second condition for stationary Stoneley waves

$$\mathbf{e}:[\mathbf{E}^+_{ep}:(\mathbf{k}^S_+ \otimes \mathbf{A}^u - \mathbf{e}\cdot\mathbf{A}^\omega) + \mathbf{D}^{\kappa,+}_p:(\mathbf{k}^S_+ \otimes \mathbf{A}^\omega)]$$
$$= \mathbf{e}:[\mathbf{E}^-_{ep}:(\mathbf{k}^S_- \otimes \mathbf{A}^u - \mathbf{e}\cdot\mathbf{A}^\omega) + \mathbf{D}^{\kappa,-}_p:(\mathbf{k}^S_- \otimes \mathbf{A}^\omega)]. \tag{76}$$

7. PHENOMENA: BOUNDARY LAYER EFFECT

To illustrate the effects in a specimen with distinct boundary effects the four point bending test under plain strain conditions is analyzed (see Fig. 5). Experimental results

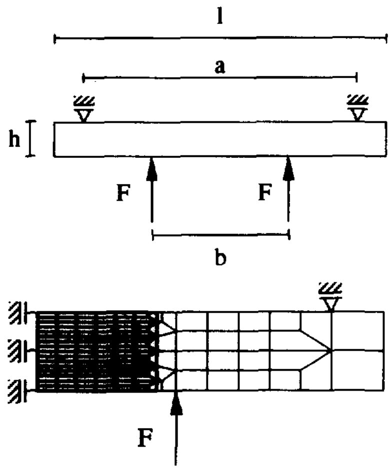

Fig. 5. Four point bending: problem and mesh discretization.

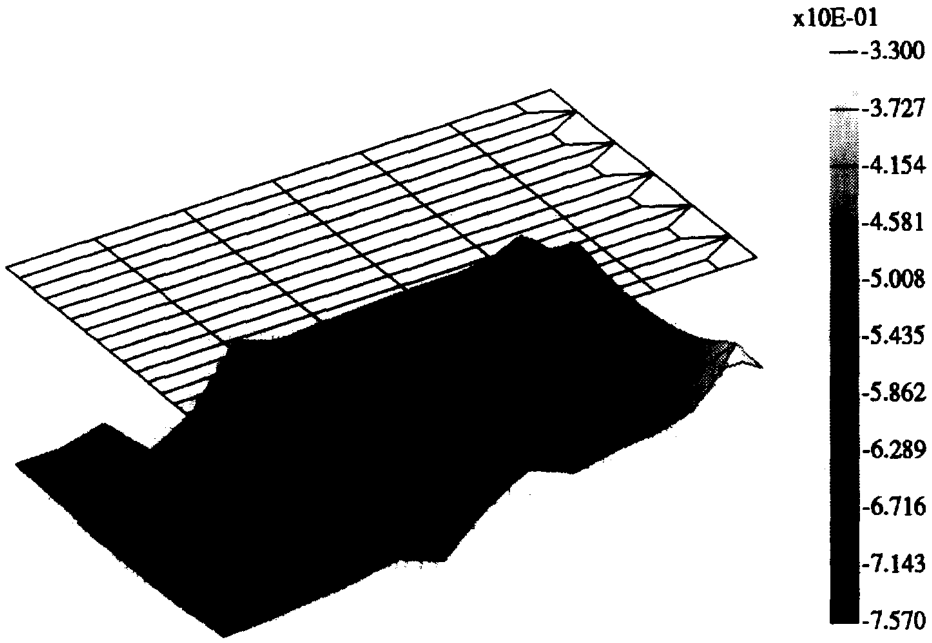

Fig. 6. Four point bending: Distribution of the generalized curvature.

were presented by Schäfer *et al.* (1990), for flexural specimens with the dimensions $l = 200$ mm, $a = 170$ mm, $b = 0.80$ mm and $h = 22.5$ mm. For simplification an augmented von Mises yield condition $F(\mathbf{s}, \boldsymbol{\mu}) = 0$ with symmetric deviatoric stress $\mathbf{s}$ is assumed (see de Borst (1990)) to describe the material behavior with an associated flow rule and perfect plasticity where the critical yield stress is $Y_0 = 310$ N/mm^2. The values of the material properties are $G_c = 0.5$ G, $l_{c,s} = 1.5$ mm, $E = 200{,}000$ N/mm^2 and $\nu = 0.3$. For the numerical simulation, nine node Cosserat elements are used with three degrees of freedom per node, whereby the deformation process is subjected to strain control.

Considering the distribution of the curvatures in Fig. 6 (only the central part of the mesh in Fig. 5 is shown), it is apparent that the harmonic disturbances appear in the plastic boundary zone which decay rapidly into the body.

These stationary waves of the curvatures are related to stationary waves of the microrotations. As a result of the analysis of body waves it is found that the direction vectors, together with the two amplitude vectors, has to form an orthogonal system for the special case of symmetric stresses $\boldsymbol{\sigma}$ (see the extension of (42) to body waves). This argument applied to Rayleigh waves by replacing $\mathbf{N}$, $\mathbf{M}^u$, $\mathbf{M}^\omega$ with $\mathbf{k}^R$, $\mathbf{A}^u$, $\mathbf{A}^\omega$ requires non-vanishing translational amplitude vectors. However, the observed phenomenon indicates that stationary rotational waves indeed appear, but stationary waves in the translational field cannot be found. So, the second condition for stationary Rayleigh waves is clearly violated.

The stationary curvature waves reflect inhomogeneous plastic deformations which were detected in the above mentioned experiments. The Lüders wedges initiate in the two outer layers of the flexural specimen as soon as the yield strength is reached during increasing external load, and they decay towards the neutral axis. It is noteworthy to mention that finite elements based on classical continuum formulations are not able to show such plastic wedge deformations which differ entirely from those in tension experiments.

8. CONCLUSIONS

For classical continua it was shown by Hadamard (1901) and others that the formulation of stationary waves and the assumption of a singularity surface of second order are identical. This could also be shown for Cosserat continua. The analysis of the conditions for stationary waves in micropolar continua leads to two conditions analogous to the two

Table 1. Localization and stationary wave conditions for Cosserat continua with symmetric σ

Body	Singularity of $\mathbf{Q}^{J1} \wedge \mathbf{e}:(\mathbf{N} \otimes \mathbf{M}^u) = 2\mathbf{M}^\omega$
Body	Singularity of $\mathbf{Q}^{B1} \wedge \mathbf{e}:(\mathbf{k} \otimes \mathbf{A}^u) = 2\mathbf{A}^\omega$
Surface	Singularity of $\mathbf{Q}^{R1} \wedge \mathbf{e}:(\mathbf{k}^R \otimes \mathbf{A}^u) = 2\mathbf{A}^\omega$
Interface	Singularity of $\mathbf{Q}^{S1} \wedge \mathbf{e}:(\mathbf{k}^S_\pm \otimes \mathbf{A}^u) = 2\mathbf{A}^\omega$

conditions when localization in the form of a singularity surface is discussed. After explicit evaluation of the components it is evident that the same requirements appear in both cases.

A simple relation between the vectors $\mathbf{N}$, $\mathbf{M}^u$, $\mathbf{M}^\omega$, respectively, $\mathbf{k}$, $\mathbf{A}^u$, $\mathbf{A}^\omega$ is possible if the restriction to a symmetric stress tensor $\boldsymbol{\sigma}$ is made. Then the second localization condition respectively the second condition for stationary waves requires the rotational vector to be orthogonal to the plane of the directional and translational vectors (see also Table 1). The conditions in Table 1 for the singularity vectors $\mathbf{M}^u$, $\mathbf{M}^\omega$ and the amplitude vectors $\mathbf{A}^u$, $\mathbf{A}^\omega$ concerning body waves are based on the fact that the only possible jumps are described by

$$[\![\mathrm{rot}\,\mathbf{u}]\!] = \mathbf{e}:[\![\boldsymbol{\varepsilon}]\!] - \mathbf{e}:\mathbf{e}\cdot\mathbf{M}^\omega = \mathbf{M}^\omega, \quad [\![\mathrm{rot}\,\boldsymbol{\omega}]\!] = \mathbf{0}. \tag{77}$$

In case of a stress state with suppresses $\mathbf{M}^\omega$, the vectors $\mathbf{N}$ and $\mathbf{M}^u$ are coaxial. Then the bifurcated vector fields $\mathbf{u}$, $\boldsymbol{\omega}$ can be derived by scalar potentials p^u and p^ω so that $\mathbf{u} = \mathbf{grad}\,p^u$ and $\boldsymbol{\omega} = \mathbf{grad}\,p^\omega$. Hence, both vector fields are laminar, only their normal components can suffer a jump. Continuous solenoidal vector fields with transversal jumps of their gradients are therefore excluded, the jumps of the divergence of the vector fields in general are not equal to zero. This is in accordance with Weingarten's first theorem (Weingarten, 1903) which correlates the longitudinal jumps of the gradient of a continuous vector field with the jumps of its divergence.

This fact is also reflected when the argument of stationary waves is considered. Here, the Hadamard theorem (Hadamard, 1901) states that longitudinal acceleration waves—identical with the existence of singular surfaces of order two—carry jumps in the expansion, whereby the vorticity remains unchanged.

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